4/4 1. (a) $\dot{x} = y$

$$\dot{y} = -V'(x) - \nu y = -4x^3 + 12x^2 - 8x - \nu y.$$

multiplying ~~the first equation~~ $\dot{x}=y$ on both sides of the second equation, we have:

$$y \cdot \dot{y} = -(4x^3+12x^2-8x)\dot{x} - \nu y^2.$$

$$\therefore \frac{1}{2}\frac{d}{dt}(y^2) = -\frac{d}{dt}(V(x)) - \nu y^2. \qquad ①$$

suppose $m=1$. then $E = \frac{1}{2}y^2 + V(x) = \frac{1}{2}y^2 + x^4 - 4x^3 + 4x^2$ is an energy of the system.

and by equation ①, we have $dE \le 0$. if $\nu > 0$.

and $dE = 0$ if $\nu = 0$.

Thus. when $\nu > 0$, the trajectories lie on the sublevel sets of the energy function, while when $\nu = 0$, the trajectories lie on the level sets of the energy function.

In both cases, due to the even leading term of the potential function, the sublevel sets and level sets of E are bounded. ~~and compact~~

Thus the trajectories are a priori bounded. in both cases.

Thus the trajectories or solution curves exist for all positive time for both $\nu = 0$ and $\nu > 0$.

4/4 (b) For equilibrium points, we have $\dot{x}=0$, $\dot{y}=0$.

~~By the first~~ Thus: $\begin{cases} y=0 \\ -V'(x) = -4x^3+12x^2-8x = 0. \end{cases}$

We have $x = 0, 1,$ or 2.

$\therefore$ The equilibrium points of the system are:

$(0,0)$, $(1,0)$ and $(2,0)$. for both $\nu=0$ and $\nu>0$.

4/4 (c) The linearization of the system is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12x^2+24x-8 & -\nu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

At (0,0). the linearization becomes:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -8 & -V\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

when $V=0$, the eigenvalues are: $\lambda_1 = 2\sqrt{2}\,i$, $\lambda_2 = -2\sqrt{2}\,i$

$\therefore$ They both ~~are~~ are pure imaginary.

when $V>0$, the eigenvalues are. $\lambda_1 = \frac{-V+\sqrt{V^2-32}}{2}$ $\lambda_2 = \frac{-V-\sqrt{V^2-32}}{2}$

since $V^2-32<V^2$, both λ_1, λ_2 have negative real parts.

At (1,0) the linearization becomes:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 4 & -V\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

when $V=0$, the eigenvalues are: $\lambda_1 = 2$ $\lambda_2 = -2$.

$\therefore$ ~~they~~ one of them have positive real part, the other negative.

when $V>0$, the eigenvalues are: $\lambda_1 = \frac{-V+\sqrt{V^2+16}}{2}$ $\lambda_2 = \frac{-V-\sqrt{V^2+16}}{2}$.

since $V^2+16>V^2$, $\lambda_1>0$. and $\lambda_2<0$.

$\therefore$ one of them have positive real part, the other negative.

At (2,0). the linearization becomes:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -8 & -V\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$ which is the same as at (0,0).

$\therefore$ when $V=0$. the eigenvalues are: $\lambda_1 = 2\sqrt{2}\,i$, $\lambda_2 = -2\sqrt{2}\,i$.

they both are pure imaginary.

when $V>0$. the eigenvalues are $\lambda_1 = \frac{-V+\sqrt{V^2-32}}{2}$, $\lambda_2 = \frac{-V-\sqrt{V^2-32}}{2}$

$\therefore$ they both have negative real parts.

4/4

(d). When $V=0$,

at (0,0), the eigenvalues are pure imaginary.

$\therefore$ we ~~can't~~ can say nothing about ~~the~~ stability or instability of the system at (0,0) by Liapunov eigenvalue theorem.

At (1,0), one of the eigenvalues have positive real part,

∴ The system is unstable at $(1,0)$

At $(2,0)$, the eigenvalues are both imaginary.

∴ We can say nothing about stability or instability of the system here.

When $V>0$,

At $(0,0)$, both eigenvalues have negative real parts.

∴ The system is asymptotically stable at this point.

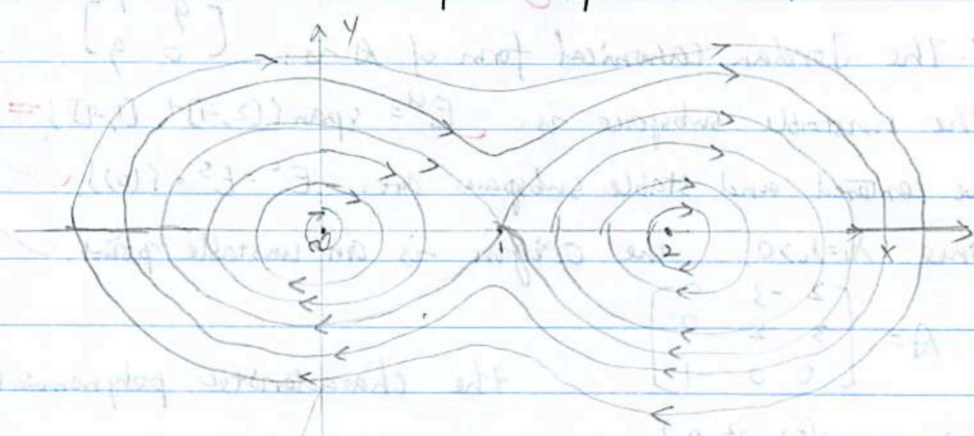
At $(1,0)$, $\lambda_1>0$ & $\lambda_2<0$.

∴ The system is unstable at this point.

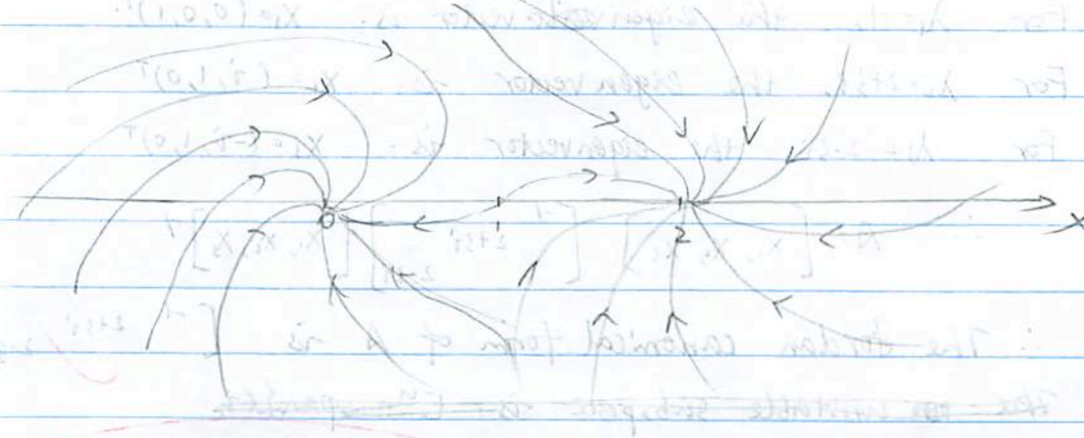
At $(2,0)$, both eigenvalues have negative real parts.

∴ The system is asymptotically stable at this point.

4/4 (e) When $V=0$, the phase portrait is:



When $V>0$, the phase portrait is:



2. 19 (a) Eigenvalues satisfy

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$$0 = \det(\lambda I - A) = \det\begin{pmatrix} \lambda-7 & 4 \\ -1 & \lambda-11 \end{pmatrix}$$

$$= \lambda^2 - 18\lambda + 81$$

$\therefore \lambda_1 = \lambda_2 = 9 > 0$;

$\therefore E^U = \mathbb{R}^2$, $E^S = E^C = \{0\}$

$$9I - A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix},$$

$\therefore \operatorname{rank}(9I - A) = 1$,

$\therefore$ The Jordan form of A is $\begin{pmatrix} 9 & 1 \\ 0 & 9 \end{pmatrix}$

The origin is an unstable point

(b). 7/7

Eigenvalues satisfy

$$0 = \det(\lambda I - A) = \det\begin{pmatrix} \lambda-2 & 3 & \\ -3 & \lambda-2 & \\ & & \lambda+1 \end{pmatrix}$$

$$= (\lambda+1)\left((\lambda-2)^2 + 9\right)$$

$\therefore \lambda_1 = -1$, $\lambda_{2,3} = 2 \pm 3i$

$(-I-A)x = 0$ ($x \in \mathbb{R}^3$) $\iff \begin{pmatrix} 3 & -3 & \\ 3 & 3 & \\ & & 0 \end{pmatrix} x = 0$

$\iff x = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$;

2.

1

$\therefore E^s = \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\};$

$$((2+3i)I - A)\, x = 0 \quad x \in \mathbb{C}^3 \iff \begin{pmatrix} -3i & -3 & \\ 3 & -3i & \\ & & -1 \end{pmatrix} x = 0$$

$$\iff x = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \cdot \beta, \quad \beta \in \mathbb{C}$$

$2 \pm 3i$ are conjugate eigenvalues,

$$\therefore E^u = \operatorname{span}\left\{ \operatorname{Re}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \operatorname{Im}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad E^c = \{0\}$$

$\therefore$ The Jordan form of A is

$$J = \begin{pmatrix} 2+3i & & \\ & 2-3i & \\ & & -1 \end{pmatrix}; \quad \text{since } A = \begin{pmatrix} 2 & -3 & \\ -3 & 2 & \\ & & -1 \end{pmatrix},$$

The origin is an unstable spiral in the x-y plane ($\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$) and a stable node on the z-axis ($\operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$)

overall the origin is unstable, and could be called a "3-D saddle"!

See pg. 99 Perko

(c). A is upper diagonal,

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$\therefore \lambda_{1,2,3} = -1, -1, 1$

$\operatorname{rank}(A+I) = \operatorname{rank}\begin{pmatrix} 0 & & \\ & 0 & 4 \\ & & 2 \end{pmatrix} = 1$;

$\therefore$ The Jordan form of A is

$$J = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

$(A+I)\overset{x \in \mathbb{R}^3}{x} = 0 \iff \begin{pmatrix} 0 & & \\ & 0 & 4 \\ & & 2 \end{pmatrix} \cdot x = 0$

$\iff x = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}$

$\therefore E^s = \left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}$

$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$(A-I)\overset{x \in \mathbb{R}^3}{x} = 0 \iff \begin{pmatrix} -2 & & \\ & -2 & 4 \\ & & 0 \end{pmatrix} \cdot x = 0$

$\iff x = \begin{pmatrix} 0 \\ 2\gamma \\ \gamma \end{pmatrix}, \quad \gamma \in \mathbb{R}$

$\therefore E^u = \left\{ \begin{pmatrix} 0 \\ 2\gamma \\ \gamma \end{pmatrix}, \gamma \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

$E^c = \{0\}$.

3.

The origin is unstable, and could be considered a "3-D saddle", as it does look like a saddle in the 2-D plane of span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

3) a) $A = \begin{bmatrix} -\alpha\beta & -1 \\ 1 & -\alpha\beta \end{bmatrix}$ ✓ $\lambda = -\alpha\beta \pm i$

5/5

For the linearized system, origin is $\begin{cases} \text{stable if } \alpha\beta \geq 0 \\ \text{unstable if } \alpha\beta < 0 \end{cases}$

5/5 b)

$r^2 = x^2 + y^2$

$\theta = \arctan(y/x)$

$$r\dot{r} = x\dot{x} + y\dot{y}$$
$$= \alpha x^2(r^4 - \beta) - xy + \alpha y^2(r^4 - \beta) + xy$$
$$= \alpha r^2(r^4 - \beta)$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x}$$
$$= \alpha xy(r^4 - \beta) + x^2 - \alpha xy(r^4 - \beta) + y^2$$
$$= r^2$$

$$\Rightarrow \begin{cases} \dot{r} = \alpha r(r^4 - \beta) = -\alpha\beta r + \alpha r^5 \\ \dot{\theta} = 1 \end{cases}$$ ✓

5/5

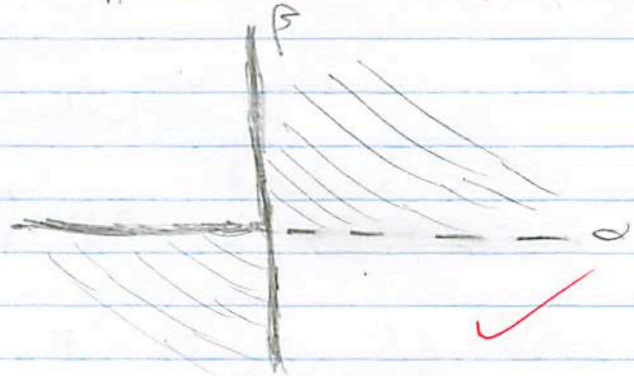
c) For sufficiently small $r > 0$, $\dot{r} < 0$ if $\alpha\beta > 0$

$\dot{r} > 0$ if $\alpha\beta < 0$

(3c p.2)

$\dot{r} = 0$ if $\alpha = 0$
$\dot{r} > 0$ if $\beta = 0, \alpha > 0$
$\dot{r} < 0$ if $\beta = 0, \alpha < 0$

$\Rightarrow$ origin is stable in the nonlinear system iff
$(\alpha \geq 0$ and $\beta > 0)$ or $(\alpha \leq 0$ and $\beta \leq 0)$



shaded = stable

3)d) Case 1 $\alpha = 0$. If $\alpha = 0$, system is a simple harmonic oscillator with a family of stable periodic orbits.

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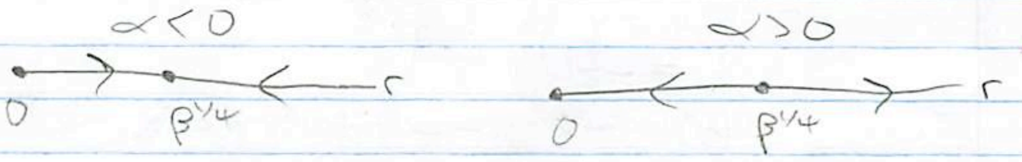
Case 2 $\alpha \neq 0$. Periodic orbit at $r = \beta^{1/4}$, provided $\beta > 0$. This p.o is stable if $\alpha < 0$, unstable if $\alpha > 0$.



Summary System has stable po's if $\alpha = 0$ or $(\alpha < 0$ and $\beta > 0)$

Ans 4:- $\dot{x} = \alpha x - y^3$

5/5

$\dot{y} = x$

$\dot{z} = \alpha z - y^2 z.$

If a point is $(a,b,c) \in \mathbb{R}^3$,

$(a,b,c) = ae_1 + be_2 + ce_3.$

$\therefore \|(a,b,c)\| \leq \|ae_1\| + \|be_2\| + \|ce_3\|$

$= |a| + |b| + |c|.$

Consider a closed 1-ball around $p = (1,2,0)$.

$X(x,y,z)$ is as given above.

$$\max_{(x,y,z)\in B_1(p)} \|X\|_{l^2} \leq \max_{(x,y,z)\in B_1(p)} |\alpha x - y^3| + \max_{(x,y,z)\in B_1(p)} |x| + \max_{(x,y,z)\in B_1(p)} |\alpha z - y^2 z|.$$

$$\max_{(x,y,z)\in B_1(p)} |\alpha x - y^3| \leq \max_{\substack{0\leq x\leq 2\\ 1\leq y\leq 3}} |\alpha x - y^3| \leq \max_{0\leq x\leq 2} |\alpha x| + \max_{1\leq y\leq 3} |y^3|$$

$$= 2|\alpha| + 27$$

$$\max_{(x,y,z)\in B_1(p)} |\alpha z - y^2 z| \leq \max_{\substack{-1\leq z\leq 1\\ 1\leq y\leq 3}} \left[|\alpha z| + |y|^2 |z|\right]$$

$$\leq |\alpha| + 9 + 1 = |\alpha| + 10.$$

$$\therefore \max_{(x,y,z)\in B_1(p)} \|X\|_{l^2} \leq 2|\alpha| + 27 + 2 + |\alpha| + 10$$

$$= 3|\alpha| + 39.$$

$\therefore$ One estimate for the time of existence $= \dfrac{1}{3|\alpha| + 39}$.

Please note that RHS of diff eq^n is polynomial in $x, y, z \Rightarrow C^\infty \Rightarrow$ Lipshitz on any bounded domain U containing $B_1(p)$.

b. Note :- The polynomial terms ~~are~~ all have odd ~~to~~ total degree.

5/5

$\therefore \bar{x}(t) = -x(t)$

$\bar{y}(t) = -y(t)$

$\bar{z}(t) = -z(t)$ ~~all~~ satisfy the diff^n.

we show explicitly.

$$\dot{\bar{x}}(t) = -\dot{x}(t) = -[\alpha x - y^3]$$

$$= \alpha(-x) - (-y)^3.$$

$$= \alpha\bar{x} - \bar{y}$$

$$\dot{\bar{y}} = -\dot{y} = -x = \bar{x}$$

$$\dot{\bar{z}} = -\dot{z} = -\alpha z + y^2 z$$

$$= \alpha(-z) - (-y)^2(-z)$$

$$= \alpha\bar{z} - \bar{y}^2\bar{z}$$

This means that reflection about the origin keeps the phase portrait unchanged.

Q. $\dot{x} = \alpha x - y^3 = f_1(x, y, z)$.

$$\left.\frac{\partial f_1}{\partial x}\right|_0 = \alpha, \quad \left.\frac{\partial f_1}{\partial y}\right|_0 = 0, \quad \left.\frac{\partial f_1}{\partial z}\right|_0 = 0.$$

$\dot{y} = x = f_2(x, y, z)$

$$\left.\frac{\partial f_2}{\partial x}\right|_0 = 1, \quad \left.\frac{\partial f_2}{\partial y}\right|_0 = 0, \quad \left.\frac{\partial f_2}{\partial z}\right|_0 = 0.$$

$\dot{z} = \alpha z - y^2 z = f_3(x, y, z)$

$$\left.\frac{\partial f_3}{\partial x}\right|_0 = 0, \quad \left.\frac{\partial f_3}{\partial y}\right|_0 = 0, \quad \left.\frac{\partial f_3}{\partial z}\right|_0 = \alpha.$$

$\therefore$ Lin sys is :-

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

eigenvalues are $\alpha, \alpha, 0$ [mat is lower triangular.].

$\alpha > 0 \Rightarrow$ ~~The~~ ~~syst~~ Non-lin system is unstable.

$\alpha \leq 0 \Rightarrow$ No comments ~~or~~ from Liap th^m.

$\alpha < 0 \Rightarrow$ " " " " " as stated theorem does not include the case where some eigenvalues are negative and some are zero.

5/5

②

d. $\Phi(\alpha,t) = (x(\alpha,t),\ y(\alpha,t),\ z(\alpha,t))^T$.

5/5

$$\frac{\partial}{\partial t} x(\alpha,t) = \alpha\, x(\alpha,t) - y^3(\alpha,t).$$

$$\therefore \frac{\partial}{\partial \alpha}\frac{\partial}{\partial t} x(\alpha,t) = \frac{\partial}{\partial \alpha}\left[\alpha \cdot x(\alpha,t)\right] - 3y^2(\alpha,t)\cdot\frac{\partial y(\alpha,t)}{\partial \alpha}$$

Let $X = \frac{\partial}{\partial\alpha}x(\alpha,t)$, $Y = \frac{\partial}{\partial\alpha}y(\alpha,t)$, $Z = \frac{\partial}{\partial\alpha}z(\alpha,t)$.

~~Now the~~ $\cancel{\begin{pmatrix}\dot x\\ \dot y\\ \dot z\\ \dot\alpha\end{pmatrix} = f\left(\begin{pmatrix}x\\ y\\ z\\ \alpha\end{pmatrix}\right)}$ ~~where f is a polynomial~~ $\cancel{f^n}$.

$\cancel{\Rightarrow f \in C^r}$

$\cancel{\therefore \frac{\partial}{\partial\alpha}\frac{\partial}{\partial t}}$

Now, we note that solution to the diff eq^n is smooth in α, t, using smoothness property and the fact that RHS of the diff eq^n is polynomial in α, x, y, z.

$$\Rightarrow \frac{\partial}{\partial t}\frac{\partial}{\partial\alpha} x(\alpha,t) = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial t} x(\alpha,t).$$

$$\therefore \frac{\partial}{\partial t} X = \alpha\cdot X + x - 3y^2 Y.$$

Similarly,

$$\frac{\partial}{\partial\alpha}\frac{\partial}{\partial t} y(\alpha,t) = \frac{\partial}{\partial t} Y = \frac{\partial}{\partial\alpha} x(\alpha,t) = X.$$

$$\frac{\partial}{\partial\alpha}\frac{\partial}{\partial t} z(\alpha,t) = \frac{\partial}{\partial t} Z = \frac{\partial}{\partial\alpha}\left[\alpha z - y^2 z\right]$$

$$= \alpha Z + z - \left[y^2 \cdot Z + z\,2y\,Y\right].$$

$$= Y(-2yz) + Z(\alpha - y^2) + z.$$

$$\therefore\ \dot{X} = \alpha X - 3y^2 Y + x.$$

$$\dot{Y} = X$$

$$\dot{Z} = Y(-2yz) + Z(\alpha - y^2) + z.$$

$$\left.\begin{array}{l} x(\alpha, 0) = 1 \\ y(\alpha, 0) = 2 \\ z(\alpha, 0) = 0 \end{array}\right\} \Rightarrow \therefore \left.\begin{array}{l} \frac{\partial x}{\partial \alpha}(\alpha, 0) = X(\alpha, 0) = 0 \\ \text{Similarly } Y(\alpha, 0) = 0 \\ Z(\alpha, 0) = 0. \end{array}\right\} \underline{\text{initial conditions.}}$$

5) a) $2\ddot{y} = \ddot{x}_1 + \ddot{x}_2$

$= 0 \qquad \left(\text{since } \ddot{x}_1 = -\ddot{x}_2\right)$

5/5 $\Rightarrow \dot{y} = \text{const.}$

b) $\ddot{r} = \frac{1}{2}(\ddot{x}_2 - \ddot{x}_1)$

$= \frac{1}{2}\left(-\frac{1}{2}(x_2 - x_1)^3 + (x_2 - x_1)\right)$

5/5 $= \frac{1}{2}\left(-\frac{1}{2} 8r^3 + 2r\right)$

$= -2r^3 + r$

$$\Rightarrow \begin{cases} \ddot{r} = r(1 - 2r^2) \\ \ddot{y} = 0 \end{cases}$$

c) Writing $\ddot{r} = -V'(r)$ with $V(r) = \frac{1}{2}r^4 - \frac{1}{2}r^2$ we conclude that

5/5
$$E = \frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^4 - \frac{1}{2}r^2$$

is conserved.

d) Write the system in first order form:

5/5
$$\begin{cases} \dot{r} = s \\ \dot{s} = r(1 - 2r^2) \\ \dot{y} = z \\ \dot{z} = 0 \end{cases}$$

Eq. pts: $s = z = 0$, $y = \text{const.}$, $r = 0, \pm\frac{1}{\sqrt{2}}$

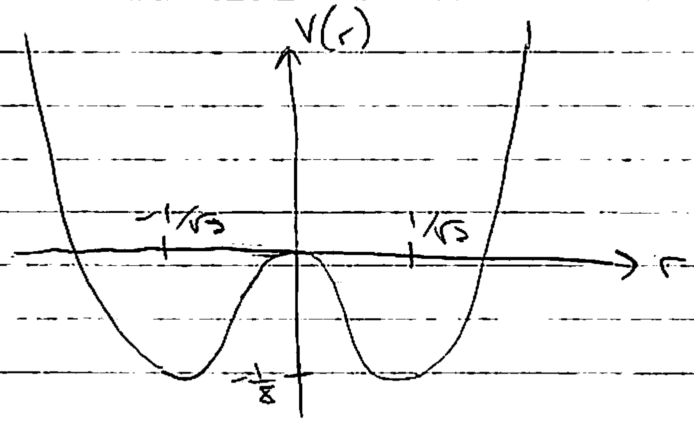
Due to decoupling of r and y, it suffices to check the stability of the system

$$\begin{cases} \dot{r} = s \\ \dot{s} = r(1 - 2r^2) \end{cases}$$

at $(r, s) = (0, 0)$, $(\frac{1}{\sqrt{2}}, 0)$, and $(-\frac{1}{\sqrt{2}}, 0)$

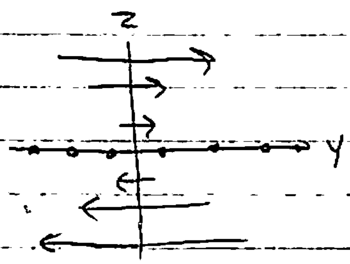
By inspection of the graph of $V(r) = \frac{1}{2}r^4 - \frac{1}{2}r^2$

(Sd p.2)



We conclude that $(r,s)=(0,0)$ is unstable while $(r,s)=(\pm 1/\sqrt{3}, 0)$ are stable, since $V(r)$ has a strict maximum at $r=0$ and strict minima at $r = \pm 1/\sqrt{3}$.

(Remark: Strictly speaking, the eq. pts. $(r,s,y,z) = (r_0, 0, \text{const.}, 0)$ with $r_0 = 0, \pm 1/\sqrt{3}$ are all unstable when viewed as a system in $\mathbb{R}^4$, since the (y,z) dynamics are governed by $\begin{cases} \dot{y} = z \\ \dot{z} = 0 \end{cases}$



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